

The University of Chicago

DIFFERENTIAL SYSTEMS WITH
BOUNDARY CONDITIONS INVOLVING
THE CHARACTERISTIC PARAMETER

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

AUGUSTO BOBONIS

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DIFFERENTIAL SYSTEMS WITH BOUNDARY CONDITIONS INVOLVING THE CHARACTERISTIC PARAMETER

1. Introduction. The present paper is concerned with the development of the theory of definitely self-adjoint systems of linear differential equations of the form

$$y_1' = \sum_{j=1}^n [A_{1j}(x) + \lambda B_{1j}(x)] y_j,$$

$$(1:1) \quad s_1[y; \lambda] = \sum_{j=1}^n [(M_{1j}^0 + \lambda M_{1j}^1) y_j(a) + (N_{1j}^0 + \lambda N_{1j}^1) y_j(b)] = 0$$

$$(a \leq x \leq b; 1, j = 1, 2, \dots, n).$$

In 1926 Bliss [1]* gave definitions of self-adjointness and definite self-adjointness for a system of the form (1:1) in which the boundary conditions are independent of λ . For such definitely self-adjoint systems he showed that the characteristic numbers are real, infinite in number, and have indices equal to their multiplicities. He also obtained some very general expansion theorems. A weakened form of this definition is given in a rather recent paper [4] by the same author. For the boundary value problems which are definitely self-adjoint according to this new definition it is true that the same expansion theorems hold, the characteristic numbers are real, and they have indices equal to their multiplicities. However, a boundary value problem may be definitely self-adjoint according to this modified definition, and not have an infinitude of characteristic values.

*The numbers in brackets refer to the bibliography at the end of this paper.

In the following pages we shall extend the definitions and results of Bliss to a system of the form (1:1) which involves the parameter λ linearly in the boundary conditions.

In Section 2 some general properties of linear systems are given. Sections 3 and 4 are devoted to the discussion of adjoint and self-adjoint systems. A definition of definite self-adjointness is given in Section 5, and fundamental theorems concerning the characteristic values of such a system, analogous to those obtained by Bliss [4] for a system whose boundary conditions are independent of λ , are also proved in this section. Sections 6 and 7 deal with auxiliary theorems and lemmas which are basic to the development of the expansion theorems given in the following section. In Section 9 sufficient conditions for the existence of an infinity of characteristic values are given. Finally, in Section 10 it is shown that a certain boundary value problem of the calculus of variations of the type studied by Reid [6] is definitely self-adjoint according to the definition here given.

The methods used in the development of this theory are much the same as those used by Bliss in [1] and [4].

2. Preliminary remarks. In the first nine sections of this paper matrix notation will be used. Square matrices of n rows and columns are denoted by capital letters. For instance, M denotes the matrix whose element in the i -th row and j -th column is M_{ij} ($i, j = 1, \dots, n$). In particular, the zero and identity matrices are denoted by O and I , respectively. Similarly, the vector (v_i) ($i = 1, \dots, n$) is denoted by the lower case letter v . If $M \equiv ||M_{ij}||$, $v \equiv (v_i)$, the vectors $(M_{ij}v_j)$ and $(v_j M_{ji})$ are denoted by Mv and vM , respectively, where the repetition of the subscript j indicates summation with respect to

this subscript over the range $1, 2, \dots, n$. For the scalar product $v_j u_j$ of the vectors $v \equiv (v_1)$, $u \equiv (u_1)$ we shall write vu . The transpose of a matrix $M \equiv \|M_{ij}\|$ is denoted by $\tilde{M} \equiv \|M_{ji}\|$. If the elements of M are differentiable functions, M' signifies the corresponding matrix of derivatives; similarly, if the components of v are differentiable functions we write $v' \equiv (v_1')$. Finally, for brevity we write $M(\lambda)$ and $N(\lambda)$ for $M^0 + \lambda M^1$ and $N^0 + \lambda N^1$, respectively, when no confusion arises.

Concerning the boundary conditions

$$s[y; \lambda] \equiv M(\lambda)y(a) + N(\lambda)y(b) = 0$$

of the system (1:1) we shall make throughout the following hypothesis:

For all values of λ the $n \times 2n$ matrix $\|M(\lambda); N(\lambda)\|$ has rank n . Moreover, there exist matrices $P(\lambda) = P^0 + \lambda P^1$, $Q(\lambda) = Q^0 + \lambda Q^1$, together with matrices M^* , N^* , P^* , Q^* independent of λ , such that the $2n \times 2n$ matrices

$$(2:1) \quad \left\| \begin{array}{cc} -P^* & -P(\lambda) \\ Q^* & Q(\lambda) \end{array} \right\|, \quad \left\| \begin{array}{cc} M(\lambda) & N(\lambda) \\ M^* & N^* \end{array} \right\|$$

are reciprocals for all values of λ .

This hypothesis is readily seen to be satisfied if there exists an $n \times n$ non-singular minor of $\|M(\lambda); N(\lambda)\|$ whose elements are independent of λ . It is also to be emphasized that the boundary conditions of our problem are unchanged if the matrices $M(\lambda)$, $N(\lambda)$ are replaced by $\Gamma(\lambda)M(\lambda)$, $\Gamma(\lambda)N(\lambda)$, respectively, where for each value of λ the matrix $\Gamma(\lambda)$ is non-singular; for our purpose, however, attention will be restricted to such matrices $\Gamma(\lambda)$ for which the resulting product matrices are linear in λ .

Since the matrices of (2:1) are reciprocals, it follows in particular that

$$(2:2) \quad M(\lambda)P(\lambda) = N(\lambda)Q(\lambda).$$

Consequently ([1], p. 562), the relations

$$(2:3) \quad t[z; \lambda] \equiv z(a)P(\lambda) + z(b)Q(\lambda) = 0$$

are boundary conditions adjoint to the boundary conditions $s[y; \lambda] = 0$.

Since (2:2) is an identity in λ , it follows by equating equal powers of λ that

$$(2:4) \quad \begin{aligned} M^0 P^0 &= N^0 Q^0, \\ M^0 P^1 + M^1 P^0 &= N^0 Q^1 + N^1 Q^0, \\ M^1 P^1 &= N^1 Q^1. \end{aligned}$$

Similarly, one has also the following relations involving M^* , N^* , P^* and Q^* :

$$(2:5) \quad \begin{aligned} -P^* M^0 - P^0 M^* &= I, & \omega^* M^0 + Q^0 M^* &= 0, \\ -P^* M^1 - P^1 M^* &= 0, & Q^* M^1 + Q^1 M^* &= 0, \\ -P^* N^0 - P^0 N^* &= 0, & Q^* N^1 + Q^1 N^* &= 0, \\ -P^* N^1 - P^1 N^* &= 0, & Q^* N^0 + Q^0 N^* &= I. \end{aligned}$$

If the matrices (2:1) are multiplied in the reverse order, then in addition to (2:2) the following further identities are obtained:

$$(2:6) \quad -M^* P(\lambda) + N^* Q(\lambda) = I,$$

$$(2:7) \quad -M(\lambda) P^* + N(\lambda) Q^* = I,$$

$$(2:8) \quad -M^* P^* + N^* Q^* = 0.$$

Now if we set

$$\begin{aligned}s^*[y] &= M^*y(a) + N^*y(b), \\ t^*[z] &= z(a)P^* + z(b)Q^*\end{aligned}$$

it follows ([1], pp. 564-565) from equations (2:5) that we have the following identity in the arguments $y(a)$, $y(b)$, $z(a)$, $z(b)$ for all values of λ

$$t^*[z]s[y; \lambda] + t[z; \lambda]s^*[y] \equiv z(x)y(x) \Big|_a^b.$$

Consequently, if we write $s[y; \lambda] \equiv s^0[y] + \lambda s^1[y]$, $t[z; \lambda] \equiv t^0[z] + \lambda t^1[z]$, it follows that

$$(2:9) \quad t^*[z]s^0[y] + t^0[z]s^*[y] \equiv z(x)y(x) \Big|_a^b,$$

$$(2:10) \quad t^*[z]s^1[y] + t^1[z]s^*[y] \equiv 0,$$

for arbitrary sets $y(a)$, $y(b)$, $z(a)$, $z(b)$. The above identities will be of fundamental importance in the development of the present work.

3. Adjoint systems of differential equations. The system (1:1) in matrix form may be written

$$(3:1) \quad y' = [A(x) + \lambda B(x)]y, \quad s[y; \lambda] = 0.$$

It will be assumed throughout that the elements of $A(x)$, $B(x)$ are real single-valued continuous functions on a finite interval $a \leq x \leq b$ and the elements of $B(x)$ are not all identically zero on this interval; moreover, that the boundary conditions $s[y; \lambda] = 0$ satisfy the hypothesis of the preceding section. The system adjoint to (3:1) is by definition

$$(3:2) \quad z' = -z[A(x) + \lambda B(x)], \quad t[z; \lambda] = 0,$$

where the adjoint boundary conditions $t[z; \lambda] = 0$ are determined

by the above equations (2:3).

The term characteristic solution will be used to denote a vector $y(x)$ whose components are not all identically zero on $a \leq x \leq b$, and which satisfies for a corresponding value λ the differential equations and two point boundary conditions of (3:1). If λ is a value for which there exist r linearly independent characteristic solutions, this value will be called a characteristic value of the system (3:1) of index r .

Let $Y(x, \lambda) = \| Y_{ij}(x, \lambda) \|$ be a fundamental matrix solution of the differential equations of (3:1); that is, the columns of $Y(x, \lambda)$ form for each value of λ a set of n linearly independent solutions of these equations. Such a fundamental matrix solution could be determined by the initial value $Y(a, \lambda) = I$. It is readily seen that such a fundamental matrix solution has its elements expressible as permanently convergent power series in λ . The determinant $D(\lambda)$ of the matrix

$$(3:3) \quad \| M(\lambda)Y(a, \lambda) + N(\lambda)Y(b, \lambda) \| = \| s_1[y_j] \|$$

is a characteristic determinant for the system (3:1) in the sense that its zeros are the characteristic values of that system. Since $M(\lambda)$ and $N(\lambda)$ are linear in λ , clearly $D(\lambda)$ is also a permanently convergent power series. It is readily seen that λ is a characteristic value of index r if and only if the matrix (3:3) has rank $n - r$. From general properties of differential systems (see Bliss [1], pp. 556-568) it also follows that if λ is a characteristic value of (3:1) of index r , then λ is also a characteristic value of the adjoint system (3:2) of index r .

Suppose now that $y(x)$ is a characteristic solution of the system (3:1) for a characteristic value λ_1 , and $z(x)$ is a

characteristic solution of the adjoint system (3:2) for a characteristic value λ_2 , $\lambda_2 \neq \lambda_1$. Then,

$$(3:1') \quad y' = (A + \lambda_1 B)y, \quad s^0[y] + \lambda_1 s^1[y] = 0,$$

$$(3:2') \quad z' = -z(A + \lambda_2 B), \quad t^0[z] + \lambda_2 t^1[z] = 0,$$

and by the use of the boundary conditions in (3:1') and (3:2'), together with the identity (2:9), it follows that

$$(\lambda_1 - \lambda_2) \int_a^b zBy \, dx = z(x)y(x) \Big|_a^b = -\lambda_1 t^*[z]s^1[y] - \lambda_2 t^1[z]s^*[y].$$

By using (2:10) the above expression can be reduced to the following equivalent conditions:

$$(3:4) \quad t^*[z]s^1[y] + \int_a^b zBy \, dx = 0,$$

$$(3:4') \quad -t^1[z]s^*[y] + \int_a^b zBy \, dx = 0,$$

$$(3:4'') \quad \frac{1}{2} [t^*[z]s^1[y] - t^1[z]s^*[y]] + \int_a^b zBy \, dx = 0.$$

Thus we have the following result.

THEOREM 3:1. If $y(x)$ is a solution of (3:1) for a value $\lambda = \lambda_1$, and $z(x)$ is a solution of (3:2) for a value $\lambda = \lambda_2$, $\lambda_1 \neq \lambda_2$, then

$$(3:4) \quad t^*[z]s^1[y] + \int_a^b zBy \, dx = 0.$$

4. Self-adjoint systems of differential equations. Following the definition of Bliss ([1], [4]) the boundary value problem (3:1) is said to be self-adjoint if the differential equations and the boundary conditions of its adjoint system are equivalent to its own for all values of λ under a transformation $z(x) = T(x)y(x)$,

where $T(x)$ is a non-singular matrix on $a \leq x \leq b$ with elements which are continuous and have continuous first derivatives on this interval. The proof of the following result is as in Bliss ([1], p. 569).

THEOREM 4:1. The system (3:1) is self-adjoint if and only if there exists a transformation matrix $T(x)$ such that

$$(4:1) \quad TA + \tilde{A}T + T' \equiv 0, \quad TB + \tilde{B}T \equiv 0,$$

$$(4:2) \quad M(\lambda)T^{-1}(a)\tilde{M}(\lambda) = N(\lambda)T^{-1}(b)\tilde{N}(\lambda).$$

The above definition of self-adjointness, together with the results of Theorem 3:1, imply the following theorem.

THEOREM 4:2. If the system (3:1) is self-adjoint with respect to a transformation matrix $T(x)$, and $y(x)$ and $y^*(x)$ are characteristic solutions corresponding to two distinct characteristic values λ and λ^* , then

$$(4:3) \quad t^*[Ty^*]s^1[y] + \int_a^b y^*S(x)y \, dx = 0,$$

where $S(x) = \tilde{T}(x)B(x)$.

This follows from equation (3:4) and the fact that $z^* = Ty^*$ is a solution of the adjoint system for $\lambda = \lambda^*$.

5. Definitely self-adjoint systems. The definition of definite self-adjointness as used by Bliss will be extended to the system here considered as follows.

The boundary value problem (3:1) is definitely self-adjoint if the following hypotheses are satisfied.

(H₁) It is self-adjoint with respect to the transformation matrix $T(x)$.

(H₂) The matrix $S(x) = \tilde{T}(x)B(x)$ is symmetric on $a \leq x \leq b$.

(H₃) The matrix

$$(5:1) \quad \begin{vmatrix} \tilde{T}(a)P^*M^1 & \tilde{T}(a)P^*N^1 \\ \tilde{T}(b)Q^*M^1 & \tilde{T}(b)Q^*N^1 \end{vmatrix}$$

belonging to the quadratic form $t^*[Ty]s^1[y] \equiv Q[y(a), y(b)] \equiv Q[y]$ is symmetric. The corresponding bilinear form $t^*[Tv]s^1[u]$ in the arguments $v(a), v(b), u(a), u(b)$ we shall denote by $Q[v(a), v(b); u(a), u(b)]$, or $Q[u; v]$ for brevity. Clearly $Q[v; v] = Q[v]$.

(H₄) The quadratic expression

$$K[y] \equiv Q[y(a), y(b)] + \int_a^b y(x)S(x)y(x) dx$$

is non-negative on $a \leq x \leq b$ for arbitrary continuous $y(x)$. The corresponding bilinear expression

$$Q[u(a), u(b); v(a), v(b)] + \int_a^b uSv dx$$

will be denoted by $K[u; v]$. Clearly $K[u; u] = K[u]$.

(H₅) For an arbitrary value λ , the only solution of the system (3:1) satisfying $K[y] = 0$ on $a \leq x \leq b$ is $y(x) \equiv 0$.

As immediate consequences of the above definition we have some important results which we state here as lemmas.

LEMMA 5:1. Hypotheses (H₂) and (H₄) imply that the quadratic form $yS(x)y$ is positive semi-definite on $a \leq x \leq b$.

For suppose that the conclusion of the lemma is not true. Then there exists some vector u and a value x_0 , $a < x_0 < b$, such that $uS(x_0)u < 0$. The continuity of the elements of $S(x)$ imply that there exists a $d > 0$ such that $uS(x)u < 0$ for $|x - x_0| < d$. If we now define $y(x) = u \xi(x)$, where $\xi(x)$ is a continuous function such that $\xi(x) \neq 0$ on $x_0 - d < x < x_0 + d$ and $\xi(x) \equiv 0$ except on this interval, we see that $yS(x)y = uS(x)u \xi^2(x) < 0$ on

$x_0 - d < x < x_0 + d$, $yS(x)y \equiv 0$ elsewhere. Then

$$K[y] = \int_{x_0-d}^{x_0+d} uS(x)u \xi^2(x) dx < 0$$

and this is a contradiction to hypothesis (H_4) .

LEMMA 5:2. Hypotheses (H_3) and (H_4) imply that the quadratic form $Q[y(a), y(b)]$ is positive semi-definite.

For suppose that there exists a vector $u(x)$ such that $Q[u] \equiv Q[u(a), u(b)] < 0$. Define a function $\xi(x)$ in the following manner: $\xi(x) \equiv 0$ on $a+d \leq x \leq b-d$, $\xi(a) = \xi(b) = 1$ and $\xi(x)$ linear on $a \leq x \leq a+d$ and $b-d \leq x \leq b$. If we set $y(x) = u(x) \xi(x)$ we obtain

$$K[y] = Q[u] + \int_a^{a+d} uSu \xi^2 dx + \int_{b-d}^b uSu \xi^2 dx.$$

Since the two integrals tend to zero as d approaches zero and $Q[u] < 0$, for d sufficiently small $y(x)$ is such that $K[y] < 0$, which is a contradiction to (H_4) . Hence Q is a positive semi-definite form.

LEMMA 5:3. If for a continuous $y(x)$ we have $K[y] = 0$, then hypotheses (H_2) , (H_3) and (H_4) imply $B(x)y(x) \equiv 0$ on $a \leq x \leq b$, $s^1[y] = 0$.

Lemmas 5:1 and 5:2 and the condition that $K[y] = 0$ imply that $ySy \equiv 0$ on $a \leq x \leq b$. This says that $Sy \equiv 0$, and therefore, since the matrix $T(x)$ is non-singular, it follows that $By \equiv 0$ on $a \leq x \leq b$. The condition $Q[y(a), y(b)] = 0$ is then a consequence of Lemma 5:2 and the relation $K[y] = 0$. If we now take the partial derivatives of the above quadratic form with respect to the arguments $y(a)$, $y(b)$, and remember that its matrix of coefficients (5:1) is symmetric, we have that $P^*s^1[y] = 0$, $Q^*s^1[y] = 0$. These two relations and the fact that the rank of $\begin{vmatrix} P^* \\ Q^* \end{vmatrix}$ is n , as

implied by equation (2:5), justify the last result of the lemma.

It is therefore seen that this lemma enables us to state that whenever hypotheses (H_2) , (H_3) and (H_4) are satisfied hypothesis (H_5) is equivalent to:

(H_5') The only solution of

$$y' = A(x)y, \quad s^0[y] = 0$$

satisfying $ySy = 0$ on $a \leq x \leq b$, $s^1[y] = 0$ is $y = 0$.

In view of Lemmas 5:1 and 5:2 it follows that $K[f] \geq 0$ for arbitrary vectors $f(x)$ whose components are bounded and integrable on $a \leq x \leq b$. In most of the following discussion the arguments of the quadratic expression K will be vectors whose components are continuous on this interval. In Sections 7 and 8, however, we will have occasion to use arguments whose elements are discontinuous. In view of the non-negativeness of K we have for arbitrary vectors $f(x)$, $g(x)$ whose components are bounded and integrable on $a \leq x \leq b$ the following inequality

$$(5:2) \quad \{K[f;g]\}^2 \leq K[f] \cdot K[g].$$

If we consider again equation (4:2) we see that its transpose, together with equation (2:2), justify relations of the form

$$(5:3) \quad \begin{aligned} \tilde{T}^{-1}(a)\tilde{M}(\lambda) &= P(\lambda)C(\lambda) \\ \tilde{T}^{-1}(b)\tilde{N}(\lambda) &= Q(\lambda)C(\lambda) \end{aligned}$$

where $C(\lambda)$ is a non-singular matrix for all values of λ . The above equations (5:3) are written to simplify the statement of the following important lemma.

LEMMA 5:4. Under the hypothesis (H₃) the matrix $C(\lambda)$ defined by equations (5:3) is independent of λ .

The proof of this lemma is as follows. Multiply equations (5:3) on the left by $-M^*$ and N^* , respectively, and add. Use of equation (2:6) then yields

$$(5:4) \quad C(\lambda) = -M^* \tilde{T}^{-1}(a) \tilde{M}(\lambda) + N^* \tilde{T}^{-1}(b) \tilde{N}(\lambda).$$

Consequently, $C(\lambda)$ is linear in λ and we may write $C(\lambda) = C^0 + \lambda C^1$.

By hypothesis (H₃) the matrix of coefficients (5:1) is symmetric, and therefore

$$(5:5) \quad \tilde{T}(a) P^* M^1 = \tilde{M}^1 P^* T(a),$$

$$(5:6) \quad \tilde{T}(b) Q^* N^1 = \tilde{N}^1 Q^* T(b),$$

$$(5:7) \quad \tilde{T}(a) P^* N^1 = \tilde{M}^1 Q^* T(b).$$

If we multiply equations (5:7) and (5:6) on the left by $-M^* \tilde{T}^{-1}(a)$ and $N^* \tilde{T}^{-1}(b)$, respectively, and add, it follows by equations (5:4), (2:8) that

$$(5:8) \quad Q^* \tilde{O}^1 = 0.$$

Multiply now equations (5:5) and (5:7) on the right by $-T^{-1}(a) \tilde{M}^*$ and $T^{-1}(b) \tilde{N}^*$, respectively, and add. An application of equations (5:4) and (2:8) shows that

$$(5:9) \quad P^* \tilde{O}^1 = 0.$$

Finally, if we multiply (5:9) and (5:8) on the left by $-M^0$ and N^0 , respectively, and add, it follows in view of relation (2:7) that $O^1 = 0$.

The following theorems are concerned with certain important general properties of definitely self-adjoint systems.

THEOREM 5:1. The characteristic values of a definitely self-adjoint boundary value problem (3:1) are real and the corresponding characteristic solutions may be chosen real.

For suppose $y = y^1 + \sqrt{-1} y^2$ were a characteristic solution for $\lambda = \lambda_1 + \sqrt{-1} \lambda_2$, $\lambda_2 \neq 0$. Then the complex conjugate $\bar{y} = y^1 - \sqrt{-1} y^2$ would be a solution for $\lambda = \bar{\lambda}$ and Theorem 4:2 would imply that $K[y, \bar{y}] = 0$. By a property of quadratic forms the above expression reduces to $K[y^1] + K[y^2] = 0$; that is, $K[y^1] = K[y^2] = 0$. By the use of Lemma 5:3 it would then follow that y^1 and y^2 were individually solutions of (3:1) for $\lambda = 0$, and since $K[y^1] = K[y^2] = 0$ we would have in view of (H_5) that $y^1 \equiv 0$, $y^2 \equiv 0$. This, however, would imply $y \equiv y^1 + \sqrt{-1} y^2 \equiv 0$, which contradicts the assumption that y was a characteristic solution. Therefore the λ 's are real and the characteristic solutions may be chosen real.

THEOREM 5:2. For a definitely self-adjoint boundary value problem (3:1) the index of a root of $D(\lambda)$ is equal to its multiplicity.

If $\lambda = \lambda_0$ is a characteristic value of index r , for this value of λ the matrix (3:3) has rank $n-r$. By a proper choice of linear combinations of solutions the columns of this matrix can be so arranged as to make $s[Y_p; \lambda_0] = 0$ ($p = 1, 2, \dots, r$). Since $D(\lambda)$ is the determinant of this matrix, at $\lambda = \lambda_0$ the derivatives of $D(\lambda)$ of order less than r will clearly vanish. The r -th derivative will have the following form

$$(5:10) \quad D^{(r)}(\lambda_0) = r! \left| s[Y_{h\lambda}; \lambda_0] + s^2[Y_h] \quad s[Y_m; \lambda_0] \right|$$

$$(h = 1, \dots, r; m = r+1, \dots, n).$$

Suppose now that the above determinant vanishes. Then there exist constants c_h, c_m ($h = 1, \dots, r; m = r+1, \dots, n$) not all zero and such that the equation

$$(5:11) \quad \sum_{h=1}^r s[Y_{h\lambda}; \lambda_0] c_h + \sum_{m=r+1}^n s[Y_m; \lambda_0] c_m = - \sum_{h=1}^r s^1[Y_h] c_h$$

is satisfied. If we set

$$\begin{aligned} y^1(x) &= \sum_{h=1}^r Y_h(x, \lambda_0) c_h, & y^2(x) &= \sum_{m=r+1}^n Y_m(x, \lambda_0) c_m \\ y^3(x) &= \sum_{h=1}^r Y_{h\lambda}(x, \lambda_0) c_h, & y(x) &= y^3(x) + y^2(x), \end{aligned}$$

it follows that

$$(5:12) \quad s[y; \lambda_0] = -s^1[y^1].$$

Now the constants c_h ($h = 1, \dots, r$) can not all be zero since the last $n - r$ columns of the determinant (5:10) are linearly independent. It follows in particular, therefore, that $y^1 \neq 0$. Since $y(x, \lambda) = \sum_{h=1}^r Y_h(x, \lambda) c_h$ satisfies (3:1) for all values of λ , it follows on differentiating with respect to λ and evaluating at $\lambda = \lambda_0$ that

$$y^{3'} = (A + \lambda_0 B)y^3 + By^1.$$

As y^2 is a solution of (3:1) for $\lambda = \lambda_0$ we have, therefore, that

$$y^1 = (A + \lambda_0 B)y + By^1, \quad s[y; \lambda_0] = -s^1[y^1].$$

Finally, as $z^1 = Ty^1$ satisfies equation (3:2) for $\lambda = \lambda_0$, it follows ([1], pp. 566-568; [6], pp. 776-777) that $K[y^1] = 0$. Hypothesis (H_5) then requires y^1 to vanish identically, which is contrary to the above established conclusion that $y^1 \neq 0$ on $a \leq x \leq b$. We have thus been led to a contradiction on the

assumption that the derivative $D^{(r)}(\lambda_0)$ is equal to zero. Hence this derivative is different from zero, and the multiplicity of λ_0 is equal to its index.

THEOREM 5:3. If for a given continuous $f(x)$ the condition

$$(5:13) \quad K[y; f] = 0$$

is satisfied by every characteristic solution $y(x)$ of the definitely self-adjoint problem (3:1), this condition also holds for every $y(x)$ which satisfies with a continuous $g(x)$ the system

$$(5:14) \quad y' = Ay + Bg, \quad s^0[y] = -s^1[g].$$

The condition (5:13) implies ([1], pp. 566-568; [6], pp. 776-777) that the non-homogeneous system

$$(5:15) \quad y' = (A + \lambda B)y + Bf, \quad s[y; \lambda] = -s^1[f]$$

has solutions for every value of λ . If $D(\lambda) \neq 0$ there is a unique solution which is of the form

$$y(x, \lambda) = Y_0(x, \lambda) + \sum_{j=1}^n Y_j(x, \lambda) c_j,$$

where $y = Y_0(x, \lambda)$ is a particular solution of the differential equations of the system (5:15). It is readily seen that the components of $y(x, \lambda)$ are expressible in the form

$$(5:16) \quad y_1(x, \lambda) = \frac{1}{D(\lambda)} \begin{vmatrix} Y_{10}(x, \lambda) & Y_{11}(x, \lambda) & \dots & Y_{1n}(x, \lambda) \\ s_1[Y_0; \lambda] + s_1^1[f] & s_1[Y_1; \lambda] & \dots & s_1[Y_n; \lambda] \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ s_n[Y_0; \lambda] + s_n^1[f] & s_n[Y_1; \lambda] & \dots & s_n[Y_n; \lambda] \end{vmatrix}.$$

In a neighborhood of a zero $\lambda = \lambda_0$ of $D(\lambda)$ the solution $y(x, \lambda)$

is well defined and analytic in λ . For in the determinant (5:16) we can add multiples of the last n columns to the first column so as to make Y_0 satisfy $s[Y_0; \lambda] = -s^1[f]$ at the root $\lambda = \lambda_0$. Furthermore, if the multiplicity of λ_0 is equal to r , the r linearly independent solutions $z(x) = z(x, \lambda_0)$ of the adjoint system (3:2) for $\lambda = \lambda_0$ afford r linearly independent sets $t^*[z]$ satisfying $t^*[z]s[Y_j; \lambda_0] = 0$ ($j = 1, 2, \dots, n$). Then r of the last n rows of (5:16) can be replaced by suitable linear combinations which vanish at $\lambda = \lambda_0$, and consequently the determinant in (5:16) has the same factor $(\lambda - \lambda_0)^r$ as does $D(\lambda)$. Hence the vector $y(x, \lambda)$ is analytic and well defined near a zero $\lambda = \lambda_0$ of $D(\lambda)$. Consequently $y(x, \lambda)$ is representable as a permanently convergent power series in λ of the form

$$(5:17) \quad y(x, \lambda) = u_0(x) + \lambda u_1(x) + \lambda^2 u_2(x) + \dots + \lambda^n u_n(x) + \dots$$

If we substitute (5:17) in (5:15) and equate coefficients of like powers of λ , it follows that

$$(5:18) \quad u_q' = Au_q + Bu_{q-1}, \quad s^0[u_q] + s^1[u_{q-1}] = 0 \quad (q = 0, 1, \dots),$$

where it is to be understood that $u_{-1} = f$. If the functions v_p are defined by $v_p = Tu_p$ ($p = -1, 0, 1, 2, \dots$) it may be shown that

$$(5:19) \quad v_p' = -v_p A - v_{p-1} B, \quad t^0[v_p] + t^1[v_{p-1}] = 0 \quad (p = 0, 1, \dots).$$

The differential equations (5:19) follow by direct substitution and use of conditions (4:1) satisfied by the transformation matrix T . The boundary conditions are a consequence of relation (4:2); in particular, use is made of the result of Lemma 5:3 concerning the independence of λ of the matrix C defined by (5:3). Equations (5:18), (5:19), (2:9) and (2:10) then lead to an expression of the form

$$\int_a^b v_p B u_{q-1} dx - \int_a^b v_{p-1} B u_q dx = u_q v_p \Big|_a^b = -t^*[v_p] s^1[u_{q-1}] + t^*[v_{p-1}] s^1[u_q];$$

that is,

$$(5:20) \quad K[u_{q-1}; u_p] = K[u_q; u_{p-1}] \quad (p, q = 0, 1, 2, \dots).$$

Now as a special case of (5:2) we have

$$(5:21) \quad \{K[u_{q+1}; u_{q-1}]\}^2 \leq K[u_{q+1}] \cdot K[u_{q-1}].$$

If we define

$$W_q = K[u_0; u_q] \quad (q = 0, 1, 2, \dots)$$

equations (5:20) and (5:21) show that

$$(5:22) \quad W_{q+p} = K[u_q; u_p], \quad W_{2q-2} W_{2q+2} \geq W_{2q}^2.$$

Now the series (5:17) converges absolutely and uniformly for

$a \leq x \leq b$, and λ in a bounded region of the complex λ -plane.

Hence we can compute $\int_a^b u_0 s y dx$ by term by term integration. In particular, the expression $K[u_0; y]$ is given by the permanently convergent power series

$$W_0 + \lambda W_1 + \lambda^2 W_2 + \dots + \lambda^n W_n + \dots.$$

It then follows that the series involving only even powers of λ ,

$$(5:23) \quad W_0 + \lambda^2 W_2 + \lambda^4 W_4 + \dots,$$

is also permanently convergent. Now some coefficient in (5:23) must vanish. For otherwise, since $W_{2q} \geq 0$ and $W_{2q-2} W_{2q+2} \geq W_{2q}^2$, it would follow that

$$\frac{W_0}{W_2} \geq \frac{W_2}{W_4} \geq \frac{W_4}{W_6} \geq \dots,$$

and for $\lambda = (W_0/W_2)^{1/2}$ the terms of (5:23) are not less than the corresponding terms of the divergent series

$$W_0 + W_0 + W_0 + \dots$$

Hence some coefficient in (5:23) is zero, and by successive applications of the inequality of (5:22) it follows that $K[u_0] = W_0 = 0$. Moreover, equation (5:18) for $q = 0$ gives

$$(5:24) \quad u_0' = Au_0 + Bf, \quad s^0[u_0] = -s^1[f].$$

Now suppose that $y = y(x)$ satisfies (5:14) with a continuous $g(x)$. Then $z = Ty$ satisfies the system

$$(5:25) \quad z' = -zA - g\tilde{T}B, \quad t^0[z] = -t^1[g\tilde{T}].$$

It then follows by the use of (2:9), (2:10) and the result of Lemma 5:3 that

$$(5:26) \quad K[y; \tilde{T}] = K[u_0; g].$$

Now the right member of (5:26) is zero because of the condition $K[u_0] = 0$ and the inequality $\{K[u_0; g]\}^2 \leq K[u_0] \cdot K[g] = 0$ implied by (5:2). Consequently, if f satisfies (5:13) with every characteristic solution y of the definitely self-adjoint system (3:1) we also have $K[y; f] = 0$ for every $y(x)$ satisfying the system (5:14) with a continuous $g(x)$.

COROLLARY 1 If $f(x)$ satisfies condition (5:13) with every characteristic solution of the definitely self-adjoint system (3:1), and in addition $f(x)$ satisfies with a continuous $g(x)$ the system

$$(5:27) \quad f' = Af + Bg, \quad s^0[f] = -s^1[g],$$

then $K[f] = 0$, $Bf \equiv 0$ on $a \leq x \leq b$, and $s^1[f] = 0$.

The first conclusion of the corollary follows immediately from the fact that $y = f(x)$ satisfies equation (5:27), which is of the same form as (5:14); the conditions $Bf \equiv 0$, $s^1[f] = 0$ are consequences of Lemma 5:3.

Since for a characteristic solution y we have $K[y] \neq 0$ by (H_5) , the different characteristic solutions corresponding to each characteristic value λ may be normed and orthogonalized. Consequently, we may denote by $\{\lambda_q\}$, $\{y_q(x)\}$ the totality of characteristic values and corresponding characteristic solutions, the latter orthonormal in the sense that

$$(5:28) \quad \begin{aligned} K[y_q; y_p] &= 1, \text{ if } q = p, \\ &= 0, \text{ if } q \neq p. \end{aligned}$$

6. The Green's matrix. Suppose that the homogeneous system

$$(6:1) \quad y' = Ay, \quad s^0[y] \equiv M^0 y(a) + N^0 y(b) = 0$$

is incompatible, that is, that $\lambda = 0$ is not a characteristic value of (3:1). Since for a definitely self-adjoint system (3:1) not all values of λ are characteristic values in view of Theorem 5:2, this condition may always be obtained by a linear change of the characteristic parameter. Then there exists a unique Green's matrix $G(x, t) = \|G_{ij}(x, t)\|$ for this system. For a detailed discussion of the Green's matrix and its properties the reader is referred to Bliss [1], Section 5. Its explicit form is

$$G(x, t) = \frac{1}{2} Y(x) \left[\frac{|x-t|}{x-t} I + D^{-1} \Delta \right] Y^{-1}(t)$$

where $D = M^0 Y(a) + N^0 Y(b)$, $\Delta = M^0 Y(a) - N^0 Y(b) = 0$, and $Y(x)$ is a fundamental matrix solution of the differential equations of the system (6:1). For $a < t < b$ the matrix $G(x, t)$ has the following

fundamental properties as a function of x :

$$(6:2) \quad G(t+0, t) - G(t-0, t) = I,$$

$$(6:3) \quad \frac{\partial}{\partial x} G(x, t) = A(x)G(x, t) \quad \text{on } a < x < t \text{ and } t < x < b,$$

$$(6:4) \quad M^0 G(a, t) + N^0 G(b, t) = 0.$$

Similarly, for $a < x < b$ this matrix has the following properties as a function of t :

$$(6:5) \quad G(x, x-0) - G(x, x+0) = I,$$

$$(6:6) \quad \frac{\partial}{\partial t} G(x, t) = -G(x, t)A(t) \quad \text{on } a < t < x \text{ and } x < t < b,$$

$$(6:7) \quad G(x, a)P^0 + G(x, b)Q^0 = 0.$$

Use will now be made of the Green's matrix to establish equivalence relations between our differential system and a certain type of integral equation.

THEOREM 6:1. For an arbitrary continuous $g(x)$ the non-homogeneous system

$$(6:8) \quad y' = Ay + Bg, \quad s^0[y] = -s^1[g]$$

is equivalent to the integral equation

$$(6:9) \quad y(x) = t^*[G(x)]s^1[g] + \int_a^b G(x, t)B(t)g(t)dt,$$

where $t^*[G(x)]$ represents $G(x, a)P^* + G(x, b)Q^*$.

Equations (6:8) and (6:6) show that

$$\frac{\partial}{\partial t} [G(x, t)y(t)] = G(x, t)B(t)g(t)$$

on $a < t < x$ and $x < t < b$. Therefore,

$$G(x, x-0)y(x) - G(x, a)y(a) = \int_a^x G(x, t)B(t)g(t) dt,$$

$$G(x, b)y(b) - G(x, x+0)y(x) = \int_x^b G(x, t)B(t)g(t) dt.$$

Addition of the two previous equations and use of the equations (2:9), (2:10), (6:5) and (6:7) leads to the relation (6:9).

Conversely, suppose $y(x)$ is defined by (6:9); that is,

$$(6:10) \quad y(x) = t^*[G(x)]s^1[g] + \int_a^x G(x, t)B(t)g(t) dt + \int_x^b G(x, t)B(t)g(t) dt.$$

Using (6:3) and (6:5) it readily follows that $y' = Ay + Bg$. Moreover, by the explicit form of $G(x, t)$ it is seen that

$$\lim_{x \rightarrow a} G(x, a) = \frac{1}{2} Y(a) [I + D^{-1}\Delta] Y^{-1}(a),$$

$$\lim_{x \rightarrow b} G(x, a) = \frac{1}{2} Y(b) [I + D^{-1}\Delta] Y^{-1}(a),$$

$$\lim_{x \rightarrow a} G(x, b) = \frac{1}{2} Y(a) [-I + D^{-1}\Delta] Y^{-1}(b),$$

$$\lim_{x \rightarrow b} G(x, b) = \frac{1}{2} Y(b) [-I + D^{-1}\Delta] Y^{-1}(b).$$

Direct substitution, together with the equation (2:7), yields

$s^0[y] = -s^1[g]$. Thus our theorem is proved.

COROLLARY 1. If $\lambda = 0$ is not a characteristic value of the boundary value problem (3:1), this system is equivalent to the integral equation

$$(6:11) \quad y(x) = \lambda t^*[G(x)]s^1[y] + \lambda \int_a^b G(x, t)B(t)y(t) dt.$$

This integral equation is not of the usual kind, since it involves outside the integral sign the end values of $y(x)$ at $x = a$ and $x = b$.

7. Auxiliary lemmas. The definition of definite self-adjointness as given in Section 5 of this paper does not furnish sufficient conditions for the existence of infinitely many characteristic values. Therefore, it will be understood in the following theorems that the range of the summation indices will be finite or infinite according as the problem has a finite or an infinite number of characteristic values. In the following it is assumed that the $y_q(x)$ ($q = 1, 2, \dots$) are the totality of characteristic solutions of a definitely self-adjoint boundary value problem (3:1). We shall also suppose that these characteristic solutions are orthonormal in the sense of (5:28).

If now for a continuous $f(x)$ generalized Fourier coefficients $c_q = c_q[f]$ are defined as $c_q = K[f; y_q]$, for an arbitrary integer m the orthonormal character of the y_q implies

$$K[f - \sum_{q=1}^m y_q c_q] = K[f] - \sum_{q=1}^m c_q^2.$$

The following lemma is, therefore, an immediate consequence of hypothesis (H_4).

LEMMA 7:1. For a continuous $f(x)$ the series $\sum_q c_q^2$ converges.

LEMMA 7:2. Suppose that the components of $h = h(x, t)$ are bounded for all values of x and t on the interval ab , and integrable in t on that interval for every fixed x ; moreover, that the components of g are continuous on this interval. If $d_q = K[y_q; g]$, $\chi_q(x) = K[y_q; h]$, where $\chi_q(x)$ is computed considering h as a function of t , then the series

$$(7:1) \quad \sum_q \chi_q(x) d_q$$

converges absolutely and uniformly on the interval $a \leq x \leq b$.

For arbitrary positive integral values p and m , let

$$\sum_{q=p}^{p+m} |\chi_q(x) d_q| = \sum \chi_\rho(x) d_\rho - \sum \chi_\sigma(x) d_\sigma,$$

where ρ takes on those integral values for which $\chi_\rho d_\rho$ is positive and σ those for which $\chi_\sigma d_\sigma$ is negative. Then

$$\begin{aligned} \sum_\rho \chi_\rho(x) d_\rho &= \sum_\rho K[h; y_\rho] d_\rho \\ &= K[h; \sum_\rho y_\rho d_\rho] \\ &\leq \left\{ K[h] \sum_\rho d_\rho^2 \right\}^{1/2} \end{aligned}$$

in view of (5:2) and the orthonormal character of the y_q . Consequently,

$$\sum_{\rho \leq p} \chi_\rho(x) d_\rho \leq R^{1/2} \left(\sum_{\rho \leq p} d_\rho^2 \right)^{1/2}$$

where R is the maximum value on $a \leq x \leq b$ of $K[h]$ computed considering h as a function of t . Similarly,

$$- \sum_{\sigma \leq p} \chi_\sigma(x) d_\sigma \leq R^{1/2} \left(\sum_{\sigma \leq p} d_\sigma^2 \right)^{1/2}.$$

Therefore,

$$\sum_{q=p}^{p+m} |\chi_q(x) d_q| \leq 2R^{1/2} \left(\sum_{q=p}^{p+m} d_q^2 \right)^{1/2},$$

and since $\sum_q d_q^2$ converges by Lemma 7:1 it follows that $\sum_q \chi_q(x) d_q$ converges absolutely and uniformly on $a \leq x \leq b$.

8. Expansion theorems. The orthonormal character of the characteristic solutions of the definitely self adjoint boundary value problem (3:1), together with the theorems proved in the preceding sections, justify the following important expansion theorems.

THEOREM 8:1. If $f(x)$ satisfies the system

$$(8:1) \quad f' = Af + Bg, \quad s^0[f] = -s^1[g]$$

with a continuous $g(x)$, then

$$K[y_q; g] = \lambda_q K[y_q; f] \quad (q = 1, 2, \dots).$$

This result is obtained immediately from (8:1), (3:2), (2:9), (2:10) and the fact that $z_q = Ty_q$ is a solution of the adjoint system for $\lambda = \lambda_q$.

THEOREM 8:2. If $f(x)$ satisfies the system (8:1) with a continuous $g(x)$, and $c_q = K[y_q; f]$, the series

$$(8:2) \quad \phi(x) = \sum_q y_q(x) c_q$$

converges absolutely and uniformly, and

$$(8:3) \quad K[y_p; f - \phi] = 0 \quad (p = 1, 2, \dots),$$

$$(8:4) \quad B(f - \phi) \equiv 0 \text{ on } a \leq x \leq b, \quad s^1[f - \phi] = 0.$$

If $d_q = K[y_q; g]$, it follows from Theorem 8:1 that $d_q = \lambda_q c_q$, and hence

$$\phi(x) = \sum_q \frac{y_q}{\lambda_q} d_q.$$

In view of the integral equation (6:11) we then have that $\phi_1(x)$, the 1-th component of $\phi(x)$, is given by a series of the form (7:1), where the corresponding vector $h(x, t)$ is the 1-th row of the matrix $G(x, t)\tilde{T}^{-1}(t)$. It then follows by Lemma 7:2 that the series representing $\phi_1(x)$ converges absolutely and uniformly on $a \leq x \leq b$.

We may then calculate $K[y_p; f - \phi]$ by term-wise evaluation. It follows that for each p ,

$$K[y_p; f - \phi] = K[y_p; f] - \sum_q K[y_p; y_q] \circ_q = K[y_p; f] - K[y_p; f] = 0.$$

This relation, together with the definition of $\phi(x)$, is seen to imply that

$$(8:5) \quad K[\phi; f - \phi] = 0.$$

An application of Theorem 5:3 now justifies the relation

$$(8:6) \quad K[f; f - \phi] = 0,$$

and by subtracting equation (8:5) from (8:6) we find that

$K[f - \phi] = 0$. By Lemma 5:3 it now follows that $B(f - \phi) = 0$ on $a \leq x \leq b$ and $s^1[f - \phi] = 0$.

COROLLARY 1. If the matrix $B(x)$ is non-singular on $a \leq x \leq b$, then for every continuous $f(x)$ satisfying (8:1) with a continuous $g(x)$ the series (8:2) converges absolutely and uniformly on $a \leq x \leq b$ and represents $f(x)$.

THEOREM 8:3. If $f(x)$ satisfies with a continuous $g(x)$ the system (8:1), and $g(x)$ in turn satisfies the system

$$(8:7) \quad g' = Ag + Bk, \quad s^0[g] = -s^1[k]$$

with a continuous $k(x)$, then the series (8:2) converges absolutely and uniformly on $a \leq x \leq b$ and represents $f(x)$.

Theorem 8:1 justifies the relation

$$(8:8) \quad \sum_q \lambda_q y_q K[y_q; f] = \sum_q y_q K[y_q; g].$$

Since $g(x)$ satisfies equation (8:7), which is similar to (8:1),

an application of the previous theorems shows that the left member of equation (8:8) converges absolutely and uniformly on $a \leq x \leq b$.

Now set $\theta(x) = \sum_q y_q(x) K[y_q; g]$. If we differentiate (8:2) with

respect to x and substitute the values of y' from (3:1), it follows that

$$(8:9) \quad \phi' = A\phi + B\theta, \quad s^0[\phi] = -s^1[\theta].$$

From systems (8:1) and (8:9) we therefore obtain

$$(8:10) \quad f' - \phi' = A(f - \phi) + B(g - \theta), \quad s^0[f - \phi] = -s^1[g - \theta].$$

Since g satisfies the system (8:7) it is a consequence of Theorem 8:2, applied to g in place of f , that $B(g - \theta) = 0$, $s^1[g - \theta] = 0$. Therefore,

$$(8:10') \quad f' - \phi' = A(f - \phi), \quad s^0[f - \phi] = 0.$$

This latter relation, in view of relations (8:4) and (H_5) , or its alternative form (H_5') , implies that $f - \phi = 0$ on $a \leq x \leq b$.

9. Sufficient conditions for the existence of infinitely many characteristic values. As pointed out in the preceding section, the definition of definite self-adjointness which has been formulated in this paper does not imply the existence of infinitely many characteristic values for a system (3:1). Here we shall give a sufficiency theorem by introducing the further hypothesis:

(H_6) For arbitrary continuous $g(x)$ the only solution of the system

$$y' = Ay + Bg, \quad s^0[y] = -s^1[g]$$

for which $K[y] = 0$ is $y = 0$ on $a \leq x \leq b$.

It is easily seen that hypothesis (H_6) implies (H_5) . Hypothesis (H_6) corresponds to a condition imposed by Bliss [1] in his original definition of definite self-adjointness.

THEOREM 9:1. If the hypotheses (H_1) , (H_2) , (H_3) , (H_4) and (H_6) are satisfied and if a continuous $f(x)$ satisfies the relation (5:13) for every characteristic solution of the definitely self-adjoint boundary value problem (3:1), then $Bf \equiv 0$ on $a \leq x \leq b$ and $s^1[f] = 0$.

The theorem follows immediately from the proof of Theorem 5:3, since there it is shown that $W_0 = K[u_0] = 0$ and thus, with the help of equation (5:18) for $q = 0$, that $Bf \equiv 0$ on $a \leq x \leq b$ and $s^1[f] = 0$.

COROLLARY 1. If $K[y] > 0$ for all non-identically vanishing continuous $y(x)$, then $f \equiv 0$ is the only vector satisfying equation (5:13) with all characteristic solutions of the boundary value problem.

COROLLARY 2. The only $f(x)$ which satisfies a system of the form (8:1) with a continuous $g(x)$ and also the relation (5:13) with every solution of the boundary value problem (3:1) is $f(x) \equiv 0$.

THEOREM 9:2. If the system (3:1) satisfies the hypotheses (H_1) , (H_2) , (H_3) , (H_4) and (H_6) it has an infinity of characteristic values.

Suppose that there are only a finite number of characteristic values, and denote by λ the number of linearly independent characteristic solutions of the system (3:1). We may also suppose that $\lambda = 0$ is not a characteristic value of this system. Since the elements of $B(x)$ are not all identically zero, continuous functions $g_\sigma(x)$ ($\sigma = 1, \dots, \lambda + 1$) can be determined so that $B(x)g_\sigma(x)$ form $\lambda + 1$ linearly independent vectors on $a \leq x \leq b$. Let $f_\sigma(x)$ denote the corresponding solution of the system (8:1) with $g(x)$ replaced by $g_\sigma(x)$. Then for arbitrary constants m_σ the vectors $f(x) = f_\sigma(x)m_\sigma$, $g(x) = g_\sigma(x)m_\sigma$ satisfy (8:1). On the other hand, values m_σ

($\sigma = 1, \dots, \lambda' + 1$) not all zero can be found so that f satisfies the relation (5:13) with each of the λ' characteristic solutions of (3:1). It would then follow by the above Corollary 2 that $f \equiv 0$ on $a \leq x \leq b$, and consequently $0 = Bg \equiv (Bg_\sigma)_{m_\sigma}$ on this interval. This, however, contradicts the above choice of the sets g_σ . Hence under the hypotheses of Theorem 9:2 the system (3:1) has infinitely many characteristic values.

10. A boundary value problem of the calculus of variations.

As pointed out in the introduction to this paper, we shall now show that an important case of the boundary value problem associated with the calculus of variations which has been discussed by Reid [6] is definitely self-adjoint according to the definition of Section 5. In the following section the symbols $\eta = (\eta_1)$, $\eta' = (\eta_1')$ will denote the functions $(\eta_1(x), \eta_2(x), \dots, \eta_n(x))$ and the set of their derivatives respectively. The tensor analysis summation convention will be used throughout.

The second variation of the problem of Bolza is of the form

$$(10:1) \quad J_2[\eta] = 2\mathcal{H}[\eta(x_1), \eta(x_2)] + \int_a^b 2\omega(x, \eta, \eta') dx,$$

where $2\mathcal{H}[\eta(x_1), \eta(x_2)]$ is a quadratic form in the $2n$ arguments $\eta_1(x_1), \eta_1(x_2)$ and

$$2\omega(x, \eta, \eta') \equiv \omega_{\eta'_i \eta'_j}(x) \eta'_i \eta'_j + 2\omega_{\eta'_i \eta_j}(x) \eta'_i \eta_j + \omega_{\eta_i \eta_j}(x) \eta_i \eta_j$$

is a quadratic form in the $2n$ variables η_1, η_1' .

An accessory minimum problem associated with the second variation is that of minimizing $J_2[\eta]$ in a class of arcs

$$\eta_1 = \eta_1(x) \qquad x_1 \leq x \leq x_2$$

satisfying conditions of the form

$$(10:2) \quad \Phi_{\alpha}(x, \gamma, \gamma') \equiv \Phi_{\alpha\gamma_j'}(x) \gamma_j' + \Phi_{\alpha\gamma_j}(x) \gamma_j = 0 \\ (\alpha = 1, 2, \dots, m < n)$$

$$(10:3) \quad \Psi_{\gamma}[\gamma(x_1), \gamma(x_2)] \equiv \Psi_{\gamma_j}^1 \gamma_j(x_1) + \Psi_{\gamma_j}^2 \gamma_j(x_2) = 0 \\ (\gamma = 1, 2, \dots, p \leq 2n)$$

and giving a fixed value to the expression

$$(10:4) \quad 2\mathcal{G}[\gamma(x_1), \gamma(x_2)] + \int_{x_1}^{x_2} \gamma_i \mathcal{K}_{ij} \gamma_j \, dx.$$

In (10:4), $2\mathcal{G}[\gamma(x_1), \gamma(x_2)]$ is a quadratic form in the $2n$ arguments $\gamma_1(x_1), \gamma_1(x_2)$. For brevity, derivatives of the quadratic forms $\mathcal{G}$ and $\mathcal{H}$ with respect to the arguments $\gamma_1(x_1), \gamma_1(x_2)$ will be indicated by $\mathcal{G}_{11}, \mathcal{G}_{12}, \mathcal{H}_{11}, \mathcal{H}_{12}$ respectively.

This accessory minimum problem is considered under the following hypotheses:

(I) The coefficients in (10:1), (10:2), (10:3) and (10:4) are all real. The functions $\omega_{\gamma_i' \gamma_j'}, \omega_{\gamma_i' \gamma_j}, \Phi_{\alpha \gamma_j'}$ are continuous and have continuous first derivatives, while the functions $\omega_{\gamma_i \gamma_j}, \Phi_{\alpha \gamma_j}, \mathcal{K}_{ij}$ are continuous on $x_1 x_2$. The matrices $\|\omega_{\gamma_i' \gamma_j'}\|$, $\|\omega_{\gamma_i \gamma_j}\|$ and $\|\mathcal{K}_{ij}\|$ are symmetric and $\|\Phi_{\alpha \gamma_j'}\|$ is of rank m at each point on this interval. Moreover, the matrix of constants $\|\Psi_{\gamma_j}^1; \Psi_{\gamma_j}^2\|$ has rank p . We also assume, as we may without loss of generality, that the rows of this latter matrix are normed and mutually orthogonal.

(II) The symmetric matrix

$$(10:5) \quad \left\| \begin{array}{cc} \omega_{\gamma_i' \gamma_j'} & \Phi_{\beta \gamma_i'} \\ \Phi_{\alpha \gamma_j'} & \mathcal{O}_{\alpha \beta} \end{array} \right\| \quad (i, j = 1, 2, \dots, n; \alpha, \beta = 1, \dots, m)$$

is non-singular on $x_1 x_2$.

An arc $\gamma = [\gamma_1(x)]$ will be called differentially admissible if the functions $\gamma_1(x)$ are continuous and have piece-wise continuous derivatives, and satisfy equations (10:2) on $x_1 x_2$. An arc whose end values satisfy equations (10:3) will be called terminally admissible. Finally, an arc which is both differentially and terminally admissible will, for brevity, be called admissible.

(III). There exist p differentially admissible arcs

$\gamma_1 = \gamma_{1\nu}(x)$ ($\nu = 1, 2, \dots, p$) such that the determinant
 $|\Psi_\gamma[\gamma_\nu(x_1), \gamma_\nu(x_2)]|$ is different from zero.

(IV). The form $2\delta[\gamma(x_1), \gamma(x_2)]$ is positive semi-definite for all terminally admissible arcs γ , and for x on $x_1 x_2$ the quadratic form $\gamma_1 \chi_{1j} \gamma_j$ is positive definite.

By defining

$$(10:6) \quad \Omega(x, \gamma, \gamma', \mu) = \omega(x, \gamma, \gamma') + \mu_\alpha \Phi_\alpha(x, \gamma, \gamma'),$$

the Euler-Lagrange differential equations and transversality conditions for the above defined accessory minimum problem can be written as

$$(10:7) \quad \frac{d}{dx} \Omega_{\gamma'_i} = \Omega_{\gamma_i} - \lambda \chi_{1j} \gamma_j, \quad \Phi_\alpha = 0,$$

$$a_{r1}^1 \{ \mathcal{H}_{1r}[\gamma] - \lambda \mathcal{H}_{1r}[\gamma] - \Omega_{\gamma'_r}(x_1) \} + a_{r1}^2 \{ \mathcal{H}_{1r}[\gamma] - \lambda \mathcal{H}_{1r}[\gamma] + \Omega_{\gamma'_r}(x_2) \} = 0$$

$$(10:8) \quad \Psi_\gamma[\gamma(x_1), \gamma(x_2)] = 0 \quad (r = 1, 2, \dots, 2n-p),$$

$$(v = 1, 2, \dots, p),$$

where $a_r = (a_{rj}^1, a_{rj}^2)$ are $2n-p$ linearly independent solutions of the equations

$$\Psi_{\gamma j}^1 a_j^1 + \Psi_{\gamma j}^2 a_j^2 = 0.$$

Without any loss of generality we may assume that the solutions a_r are normed and mutually orthogonal. Consequently, the $2n \times 2n$ matrix

$$\begin{vmatrix} \Psi_{rj}^1 & \Psi_{rj}^2 \\ a_{rj}^1 & a_{rj}^2 \end{vmatrix}$$

is orthogonal and we have therefore

$$\begin{aligned} a_{rj}^1 a_{r1}^1 + \Psi_{rj}^1 \Psi_{r1}^1 &= I_{j1}, \\ a_{rj}^1 a_{r1}^2 + \Psi_{rj}^1 \Psi_{r1}^2 &= 0_{j1}, \\ a_{rj}^2 a_{r1}^1 + \Psi_{rj}^2 \Psi_{r1}^1 &= 0_{j1}, \\ a_{rj}^2 a_{r1}^2 + \Psi_{rj}^2 \Psi_{r1}^2 &= I_{j1}. \end{aligned} \quad (10:9)$$

The non-singularity of the matrix (10:5) insures that the system

$$\Phi_\alpha(x, \eta, \eta') = 0, \quad \Omega_{\eta'_i}(x, \eta, \eta', \mu) = \zeta_i$$

has solutions

$$\eta_1' = \Pi_1(x, \eta, \zeta), \quad \mu_\alpha = U_\alpha(x, \eta, \zeta)$$

which are linear in the variables η_1, ζ_1 and with coefficients which are continuous functions of x on $x_1 x_2$. Consequently, the differential equations (10:7) take the canonical form

$$\begin{aligned} \eta_1' &= \mathcal{A}_{1j}(x) \eta_j + \mathcal{B}_{1j}(x) \zeta_j, \\ \zeta_1' &= \mathcal{C}_{1j}(x) \eta_j - \mathcal{A}_{j1}(x) \zeta_j - \lambda \mathcal{K}_{1j}(x) \eta_j. \end{aligned} \quad (10:10)$$

The coefficients of (10:10) are continuous, the matrices $\|\mathcal{B}_{1j}\|$, $\|\mathcal{C}_{1j}\|$ are symmetric, and $\|\mathcal{B}_{1j}\|$ has rank $n-m$ on the interval $x_1 x_2$. The boundary value problem consisting of (10:10) and the linear boundary conditions (10:8) has been shown to be self-adjoint

([5],[6]) under the transformation matrix

$$T = \begin{vmatrix} 0_{1j} & I_{1j} \\ -I_{1j} & 0_{1j} \end{vmatrix}$$

The corresponding matrix S is therefore

$$S = \begin{vmatrix} \chi'_{1j} & 0_{1j} \\ 0_{1j} & 0_{1j} \end{vmatrix}$$

which is symmetric since the matrix $\|\chi'_{1j}\|$ is symmetric. Conditions (H_1) and (H_2) of the definition of definitely self-adjoint problems are thus satisfied for the system (10:10), (10:8).

Consider again the boundary conditions (10:8). By a property of quadratic forms ([2], pp. 11-12) this system can be written as

$$(10:11) \quad \begin{aligned} & \{ \mathcal{H}_{11}[a_r] - \lambda \mathcal{G}_{11}[a_r] \} \eta_1(x_1) - a_{r1}^1 \zeta_1(x_1) \\ & + \{ \mathcal{H}_{12}[a_r] - \lambda \mathcal{G}_{12}[a_r] \} \eta_1(x_2) + a_{r1}^2 \zeta_1(x_2) = 0, \end{aligned}$$

$$\Psi_\gamma [\eta(x_1), \eta(x_2)] = 0 \quad (\gamma=1, \dots, p; r=1, 2, \dots, 2n-p).$$

It is seen that the above boundary conditions are of the same form as those in the system (3:1) for the sets η_1, ζ_1 , but the matrices $M(\lambda), N(\lambda)$ are now $2n$ -dimensional. We will now transform these boundary conditions to a more convenient form. Multiply the matrix of coefficients of the equation (10:11) on the left by the non-singular matrix

$$(10:12) \quad \begin{vmatrix} I_{rs} & \lambda \mathcal{G}[a_r; \Psi_\gamma] \\ 0_{\gamma s} & I_{\gamma\nu} \end{vmatrix}$$

($\gamma, \nu=1, 2, \dots, p; r, s=1, \dots, 2n-p$).

In (10:12) it is to be understood that $\mathcal{H}[a_r; \Psi_\gamma]$ is the bilinear expression $\mathcal{H}_{11}[a_r] \Psi_{\gamma i}^1 + \mathcal{H}_{12}[a_r] \Psi_{\gamma i}^2$. Then we obtain for the matrices $M(\lambda)$ and $N(\lambda)$ the following expressions, in which it is to be understood that i, r , and γ are row indices and j, s , and ν are column indices, and the range of these indices are as indicated previously.

$$M(\lambda) = \left\| \begin{array}{cc} \mathcal{H}_{11}[a_r] - \lambda \mathcal{H}_{j1}[a_r] + \lambda \mathcal{H}[a_r; \Psi_\nu] \Psi_{\nu j}^1 & -a_{rj}^1 \\ \Psi_{\gamma j}^1 & 0_{\gamma j} \end{array} \right\|,$$

$$N(\lambda) = \left\| \begin{array}{cc} \mathcal{H}_{j2}[a_r] - \lambda \mathcal{H}_{j2}[a_r] + \lambda \mathcal{H}[a_r; \Psi_\nu] \Psi_{\nu j}^2 & -a_{rj}^2 \\ \Psi_{\gamma j}^2 & 0_{\gamma j} \end{array} \right\|.$$

The transformation (10:12) obviously does not change the homogeneous boundary conditions (10:11).

An application of equations (2:2) and (4:2) give us expressions for $P(\lambda)$ and $Q(\lambda)$ as follows:

$$P(\lambda) = \left\| \begin{array}{cc} -a_{s1}^1 & 0_{1\nu} \\ \mathcal{H}_{i1}[a_s] - \lambda \mathcal{H}_{i1}[a_s] + \lambda \mathcal{H}[a_s; \Psi_\gamma] \Psi_{\gamma i}^1 & \Psi_{\gamma i}^1 \end{array} \right\|,$$

$$Q(\lambda) = \left\| \begin{array}{cc} -a_{s1}^2 & 0_{1\nu} \\ \mathcal{H}_{i2}[a_s] - \lambda \mathcal{H}_{i2}[a_s] + \lambda \mathcal{H}[a_s; \Psi_\gamma] \Psi_{\gamma i}^2 & \Psi_{\gamma i}^2 \end{array} \right\|.$$

With the help of equations (10:9) we can determine the matrices M^* , N^* , P^* , Q^* independent of λ in such a way that the

matrices (2:1) are reciprocals. They can be chosen having the following form

$$P^* = \begin{vmatrix} 0_{1s} & -\Psi_{\gamma i}^1 \\ a_{s1}^1 & -\mathcal{N}_i[\Psi_\gamma] \end{vmatrix}, \quad Q^* = \begin{vmatrix} 0_{1s} & \Psi_{\gamma i}^2 \\ a_{s1}^2 & -\mathcal{N}_{iz}[\Psi_\gamma] \end{vmatrix},$$

$$M^* = \begin{vmatrix} -a_{rj}^1 & 0_{rj} \\ \mathcal{N}_{ji}[\Psi_\gamma] & -\Psi_{\gamma j}^1 \end{vmatrix}, \quad N^* = \begin{vmatrix} -a_{rj}^2 & 0_{rj} \\ \mathcal{N}_{jz}[\Psi_\gamma] & \Psi_{\gamma j}^2 \end{vmatrix}.$$

A rather lengthy computation and use of equations (10:9) yield for the matrix of coefficients (5:1) of the form Q for this special problem the matrix

$$(10:13) \quad \begin{vmatrix} a_{r1}^1 \mathcal{H}[a_r; a_s] a_{s1}^1 & 0_{1j} & a_{r1}^1 \mathcal{H}[a_r; a_s] a_{s1}^2 & 0_{1j} \\ 0_{1j} & 0_{1j} & 0_{1j} & 0_{1j} \\ a_{r1}^2 \mathcal{H}[a_r; a_s] a_{s1}^2 & 0_{1j} & a_{r1}^2 \mathcal{H}[a_r; a_s] a_{s1}^2 & 0_{1j} \\ 0_{1j} & 0_{1j} & 0_{1j} & 0_{1j} \end{vmatrix},$$

where $\mathcal{H}[a_r; a_s] = \mathcal{H}[a_s; a_r] = a_{r1}^1 \mathcal{H}_{11}[a_s] + a_{r1}^2 \mathcal{H}_{12}[a_s]$. The matrix (10:13) is seen to be symmetric; that is, hypothesis (H_3) of the definition of definite self-adjointness is satisfied.

Corresponding to arbitrary values $\eta_1(x_1)$, $\eta_1(x_2)$ the equations

$$\eta_1(x_1) = c_r a_{r1}^1 + d_\gamma \Psi_{\gamma i}^1$$

$$\eta_1(x_2) = c_r a_{r1}^2 + d_\gamma \Psi_{\gamma i}^2$$

have the unique solution

$$c_r = a_{r1}^1 \eta_1(x_1) + a_{r1}^2 \eta_1(x_2),$$

$$d_\gamma = \Psi_{\gamma 1}^1 \eta_1(x_1) + \Psi_{\gamma 1}^2 \eta_1(x_2) \quad \begin{matrix} (\gamma = 1, \dots, p; \\ r = 1, 2, \dots, 2n-p). \end{matrix}$$

In particular, for a terminally admissible arc the constants d_γ are zero. For general sets $\eta_1(x_1)$, $\zeta_1(x_1)$, $\eta_1(x_2)$, $\zeta_1(x_2)$ the form Q of (H_3) for this problem turns out to be the quadratic form $2\mathcal{Q}$ with the corresponding arguments $c_r a_{r1}^1$, $c_r a_{r1}^2$. As $\Psi_{\gamma j}^1 a_{rj}^1 + \Psi_{\gamma j}^2 a_{rj}^2 = 0$ ($\gamma = 1, \dots, p$; $r = 1, 2, \dots, 2n-p$), hypothesis (IV) is seen to imply condition (H_4) for this boundary value problem. Finally, the above hypothesis (III) may be shown to be equivalent to the condition that there is no characteristic solution η_1 , ζ_1 of this boundary value problem for which $\eta_1 \equiv 0$ on $x_1 x_2$ (see, for example Bliss [3], p. 48). Consequently hypothesis (H_5) is also satisfied and the above defined accessory boundary value problem has been shown to be definitely self-adjoint according to the definition of Section 5.

The above hypotheses on the boundary value problem considered in this section are more special than those of Reid [6], in that we have assumed the positiveness of the quadratic form $G[\eta(a), \eta(b)]$ and the positive definiteness of $\eta_1 K_{1j} \eta_j$. On the other hand, in another sense our conditions are weaker in that we have only assumed the non-singularity of the matrix $(10;5)$, whereas the hypotheses of Reid imply that $\omega_{\eta_i \eta_j'}(x) \pi_1 \pi_j > 0$ for all sets $(\pi_j) \neq (0_j)$ satisfying $\Phi_{\alpha \eta_j'}(x) \pi_j = 0$ ($\alpha = 1, \dots, m$).

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VITA

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